Announcements

1) HW 2 – Webwork portion due Friday, rest due Tuesday

2) Quiz next Thursday, covering today's and Monday's material

General Set-up for integrating factors

Start with a linear equation

$$a_1(x)\frac{dy}{dx} + a_2(x)y = a_3(x).$$

Put this in standard form by dividing both sides by $a_1(x)$:

We get

$$\frac{dy}{dx} + \frac{a_2(x)}{a_1(x)} y = \frac{a_3(x)}{a_1(x)}$$

(watch out for $a_1(x) = 0$!)

Let $P(x) = \frac{a_2(x)}{a_1(x)}$,

$Q(x) = \frac{a_3(x)}{a_1(x)}$. We're

solving $\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$

Multiply by the integrating factor $\mu(x)$.

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)Q(x)$$

What is $\mu(x)$?

We want the left hand side to be the derivative of a product.

Need: $\mu'(x) = \mu(x)P(x)$

Then

$$\frac{\mu'(x)}{\mu(x)} = P(x),$$

and integrating,

$$\ln \mu(x) = \int P(x)\,dx,$$

So

$$\boxed{\mu(x) = e^{\int P(x)\,dx}}$$

Now to solve the problem, integrate both sides of

$$\mu(x)Q(x) = \underbrace{\mu(x)\frac{dy}{dx} + \mu(x)P(x)y}_{= \frac{d}{dx}(\mu(x)y)}$$

by our choice of $\mu(x)$.

So

$$\boxed{\int \mu(x)Q(x)\,dx = \mu(x)y + C}$$

Example 1 : Solve

$$t^2 \frac{dx}{dt} + 3tx = t^4 \ln(t) + 1$$

if $x(1) = 4/3$

First, divide by t^2 to get

$$\frac{dx}{dt} + \frac{3}{t} x = t^2 \ln(t) + \frac{1}{t^2}$$

$P(t) = \frac{3}{t}$, $Q(t) = t^2 \ln(t) + \frac{1}{t^2}$

By the general formula,

$$\mu(t) = e^{\int P(t)\,dt}$$

$$= e^{\int \frac{3}{t}\,dt}$$

$$= e^{3\int \frac{1}{t}\,dt}$$

$$= e^{3\ln(t)}$$

$$= e^{\ln(t^3)}$$

$$= \boxed{t^3}$$

To solve

$$\frac{dx}{dt} + \frac{3}{t}x = t^2\ln(t) + \frac{1}{t^2},$$

multiply both sides by the integrating factor $\mu(t) = t^3$.

$$t^3\frac{dx}{dt} + 3t^2x = t^5\ln(t) + t$$

$$= \frac{d}{dt}\left(t^3x\right) = t^5\ln(t) + t$$

Integrating both sides
with respect to t,

$$\underbrace{\int \frac{d}{dt}(t^3 x)\,dt}_{= t^3 x + C} = \int (t^5 \ln(t) + t)\,dt$$

$$= t^3 x + C = \int (t^5 \ln(t) + t)\,dt$$

Just have to integrate the
right-hand side.

$$\int (t^5 \ln(t) + t)\,dt$$

$$= \int t^5 \ln(t)\,dt + \underbrace{\int t\,dt}_{= \frac{t^2}{2}}$$

$\int t^5 \ln(t)\,dt$ is integration by parts:

$u = \ln(t) \qquad v = t^6/6$

$du = \frac{1}{t}\,dt \qquad dv = t^5\,dt$

Then the integral is

$$\frac{t^6}{6}\ln(t) - \int \frac{t^6}{6} \cdot \frac{1}{t}\,dt$$

$$= \frac{t^6}{6}\ln(t) - \frac{t^6}{36}.$$

Putting all this together,

$$t^3 x + C = \frac{t^6}{6}\ln(t) - \frac{t^6}{36} + \frac{t^2}{2},$$

So

$$x = \frac{\frac{t^6}{6}\ln(t) - \frac{t^6}{36} + \frac{t^2}{2} - C}{t^3}.$$

Since $x(1) = 4/3$,

$$4/3 = \frac{1}{2} - \frac{1}{36} + C$$

$$C = \frac{48}{36} - \frac{18}{36} + \frac{1}{36}$$

$$= 31/36$$