

Announcements

- 1) HW 2 - Webwork portion due Friday, rest due Tuesday
- 2) Quiz next Thursday, covering today's and Monday's material

General Set-up for integrating factors

Start with a linear equation

$$a_1(x) \frac{dy}{dx} + a_2(x)y = a_3(x).$$

Put this in **standard form**

by dividing both sides by

$a_1(x)$:

We get

$$\frac{dy}{dx} + \frac{a_2(x)}{a_1(x)} y = \frac{a_3(x)}{a_1(x)}$$

(watch out for $a_1(x) = 0$!)

$$\text{Let } P(x) = \frac{a_2(x)}{a_1(x)},$$

$$Q(x) = \frac{a_3(x)}{a_1(x)}. \text{ We're}$$

Solving $\frac{dy}{dx} + P(x)y = Q(x)$

Multiply by the
integrating factor $\mu(x)$.

$$\mu(x) \frac{dy}{dx} + \mu(x) P(x)y = \mu(x) Q(x)$$

What is $\mu(x)$?

We want the left hand side to
be the derivative of a product.

$$\text{Need: } \mu'(x) = \mu(x) P(x)$$

Then

$$\frac{u'(x)}{u(x)} = P(x),$$

and integrating,

$$\ln u(x) = \int P(x) dx,$$

So

$$u(x) = e^{\int P(x) dx}$$

Now to solve the problem, integrate both sides of

$$\mu(x)Q(x) = \underbrace{\mu(x) \frac{dy}{dx} + \mu(x)P(x)y}$$

$$= \frac{d}{dx} (\mu(x)y)$$

by our choice of $\mu(x)$.

So

$$\int \mu(x)Q(x)dx = \mu(x)y + C$$

Example 1 : Solve

$$t^2 \frac{dx}{dt} + 3tx = t^4 \ln(t) + 1$$

$$\text{if } x(1) = 4/3$$

First, divide by t^2 to get

$$\frac{dx}{dt} + \frac{3}{t}x = t^2 \ln(t) + \frac{1}{t^2}$$

$$P(t) = \frac{3}{t}, \quad Q(t) = t^2 \ln(t) + \frac{1}{t^2}$$

By the general formula,

$$u(t) = e^{\int P(t) dt}$$

$$= e^{\int \frac{3}{t} dt}$$

$$= e^{3 \int \frac{1}{t} dt}$$

$$= e^{3 \ln(t)}$$

$$= e^{\ln(t^3)}$$

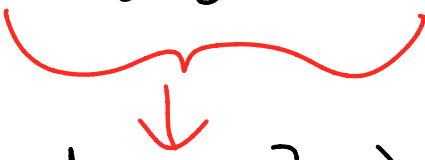
$$= \boxed{t^3}$$

To solve

$$\frac{dx}{dt} + \frac{3}{t}x = t^2 \ln(t) + \frac{1}{t^2},$$

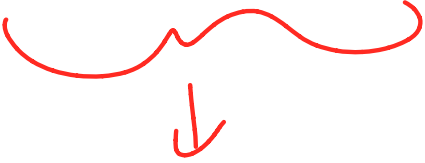
multiply both sides by
the integrating factor $\mu(t) = t^3$.

$$t^3 \frac{dx}{dt} + 3t^2 x = t^5 \ln(t) + t$$


$$= \frac{d}{dt} (t^3 x) = t^5 \ln(t) + t$$

Integrating both sides
with respect to t ,

$$\int \frac{d}{dt} (t^3 x) dt = \int (t^5 \ln(t) + t) dt$$


$$= t^3 x + C = \int (t^5 \ln(t) + t) dt$$

Just have to integrate the
right-hand side.

$$\int (t^5 \ln(t) + t) dt$$

$$= \int t^5 \ln(t) dt + \underbrace{\int t dt}_{= \frac{t^2}{2}}$$

$\int t^5 \ln(t) dt$ is integration

by parts:

$$u = \ln(t)$$

$$v = \frac{t^6}{6}$$

$$du = \frac{1}{t} dt$$

$$dv = t^5 dt$$

Then the integral is

$$\frac{t^6}{6} \ln(t) - \int \frac{t^6}{6} \cdot \frac{1}{t} dt$$
$$= \frac{t^6}{6} \ln(t) - \frac{t^6}{36}.$$

Putting all this together,

$$t^3 x + C = \frac{t^6}{6} \ln(t) - \frac{t^6}{36} + \frac{t^2}{2} + 1$$

So

$$X = \frac{\frac{t^6}{6} \ln(t) - \frac{t^6}{36} + \frac{t^2}{2} - C}{t^3}$$

Since $X(1) = 4/3$,

$$4/3 = \frac{1}{2} - \frac{1}{36} + C$$

$$C = \frac{48}{36} - \frac{18}{36} + \frac{1}{36}$$

$$= 31/36$$